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Citation	1
DOI	https://doi.org/10.1007/978-3-030-72322-4
Publisher	Springer Nature
Download date	2026-04-17 21:10:38
Link to Item	http://hdl.handle.net/20.500.14131/890

Powerful Mathematica Codes for Goodness-of-Fit Tests for Censored Data

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Abstract

In reliability studies of energy and electrical systems life data are often censored, because life tests are terminated, and life data are analyzed before the failure of all sample units. The most important task to accomplish a successful reliability analysis is to choose, through statistical goodness-of-fit tests, the correct or nearly correct probability distribution to describe the failure mechanism of given experimental data. However, due to censoring, this task would not be as easy as testing complete samples. Unfortunately, the built-in functions and codes of the available computation programs are not valid to test for incomplete or censored samples and give completely wrong results if they are used for that purpose, even on the most sophisticated ones like Mathematica and MATLAB. On the other hand, there is a high chance to slip up when trying to perform this type of tests by someone with humble probabilistic and mathematical background. Correct performance of such tests requires a deep knowledge in how to treat the estimating equations of the candidate distribution's parameters from a censored sample. This type of equations is usually implicit, which often needs a careful numerical treatment to be successfully solved. Also, we should keep in mind that the test statistics formulas of censored samples are different from those of complete samples. The corresponding critical value of the test must be modified according to the type of the distribution nominated, the degree of censoring, and the complete sample size. Therefore, there is a crucial need to have codes that safely run the tests and give reliable results. This book chapter is devoted to introducing efficient Mathematica codes for two of the best goodness-of-fit tests for censored data, the Cramér–von Mises and Anderson-Darling tests for Weibull and lognormal distributions, which are useful in a great variety of applications in energy studies, particularly as models for product life. The codes are presented together with some practical examples extracted from the literature in various topics of energy systems and related fields.

Keywords: Mathematica, MATLAB, Goodness-of-fit tests, Cramér–von Mises test, Anderson-Darling test, Probability plots, Codes, Reliability, Probability Distributions.

1. Introduction and fundamentals

Life tests or accelerated life tests of products and material are used to produce life data under working conditions or under more severe conditions. The produced lives are then extrapolated to the working conditions. Such tests start with a fixed number of objects, however, waiting for all objects to fail might take a long time that would be costly or impossible to be achieved. For instance, in [1] the authors estimated the mean time to failure of a certain type of solar cells to be around 300 years. The researchers used accelerated life test by applying several stresses on a group of cells, to get information quickly on their life distribution. Yet, they predetermined the time of the test and built their estimates only based on the observed failures. This procedure is referred to as type I right censored sampling [2]. After running the life tests, we obtain an experimental data of lifetimes. The data is then fitted to a probability distribution that provides a reasonable approximation to the data's actual distribution. Distribution fitting is the task of selecting an appropriate probabilistic distribution from a set of candidate distributions, given data. This task is usually accomplished using analytical tests, called goodness-of-fit tests. Analytical tests can be supported by graphical methods to get an initial impression of which distribution is acceptable and which is not. The analytical goodness-of-fit tests are statistical hypothesis tests used to decide if an experimental or observational sample comes from a population with a specific distribution. There are multiple types of goodness-of-fit tests and detailed reviews can be found in [3]. However, in this chapter, we will focus on only two of them that we believe more convenient than other available goodness-of-fit tests, namely, the Cramér–von Mises (CVM) [4] and Anderson-Darling (AD) [5] goodness-of-fit tests. The reasons for choosing these two specific tests are as follows:

1) They are less sensitive than other tests like the chi-square goodness-of-fit test [6], which is sensitive to how the binning of the data is performed, and thus needs human intervention making the decision of the test subjective, which differs from one person to another. Moreover, this type of tests demands a relatively large sample size to give correct results.
2) They are less extreme (or rigorous) than some other tests like Kolmogorov-Smirnov test [7], whose test statistics is the supremum of all vertical distances between the empirical and theoretical distribution functions. Therefore, the test is more sensitive near the center of the distribution than at its tails and may lead to the rejection of a theoretical distribution when only one of the distances is significantly large even if all other distances are reasonable.
3) They are applicable to non-normal distributions as opposed to some other tests like Jarque-Bera [8] test.
4) They form a strong team when both considered in making the decision. This is because AD gives more weight to the tail, whereas CVM gives more weight to the center of the

distribution [3, page 110]. Therefore, together would give a satisfactory picture about the closeness of the theoretical and empirical distributions.

In their basic forms, the CVM and AD tests assume that the distribution under consideration is normal, and its parameters are known. However, in practical life experiments, the life distribution of experimental data sets is usually skewed and thus, not normal Also, the parameters of the assumed life distribution are often unknown and estimated from the given data. More flexible distributions are preferred like the Weibull and lognormal distributions considered in this chapter. These reasons necessitate modifying their basic test statistics or their critical values. This is not an easy task for practitioners from different fields, who just need to utilize statistical testing to choose the best model for their experimental data without studying the history of the tests, their tedious mathematical details and assumptions, and the modifications of their critical values when changing the distribution tested.

Mathematically speaking, both CVM and AD test statistics belong to the class of quadratic statistics [3, page 100] having the following general form

$$Q = n \int_{-\infty}^{\infty} \{F_n(x) - F(x)\}^2 \psi(x) dF(x). \tag{1}$$

In equation (1), n is the observed sample size, $F_n(x)$ is the cumulative distribution function (CDF) of the sample, which is usually called the empirical CDF, $F(x)$ is the CDF of the underlying theoretical distribution, and $\psi(x)$ is a suitable function that gives weights to the squared difference $\{F_n(x) - F(x)\}^2$. When $\psi(x) = 1$, the statistics Q becomes the Cramér–von Mises statistics that can be easily reduced to

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - z_i\right)^2, \tag{2}$$

and when $\psi(x) = \{F(x) - F^2(x)\}^{-1}$, Q becomes the Anderson-Darling statistics simplified to

$$AD = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1)\big(Log(z_i) + Log(1 - z_{n+1-i})\big) . \tag{3}$$

In equations (2) and (3), $z_i = F(x_i)$, $i = 1,2,\dots,n$, are the ranks of the observed values x_i under $F(x)$, where $x_1 \leq x_2 \leq \cdots \leq x_n$ are assumed ordered.

Formulas (2) and (3) are the test statistics for complete samples of lives, when all the items tested have failed before the end of the test. The corresponding modified formulas for censored samples, in particular type I right censored samples, are available in many references but were first introduced in [9] and given respectively by

$$CVM = \sum_{i=1}^{r}\left(\frac{2i-1}{2n} - z_i\right)^2 - \frac{r(4r^2-1)}{12n^2} + np\left(\frac{r^2}{n^2} - p\frac{r}{n} + \frac{1}{3}p^2\right), \tag{4}$$

$$AD = \frac{1}{n}\sum_{i=1}^{r}(2i-1)\big(Log(1-z_i) - Log(z_i)\big) - 2\sum_{i=1}^{r} Log(1-z_i)$$

$$+n\left(\frac{2r}{n} - \frac{r^2}{n^2} - 1\right)Log(1-p) + \frac{r^2}{n}Logp - np, \tag{5}$$

where r and $p = r/n$ are the size and percentage of the censored sample, respectively. In life studies, type I right censoring is the most commonly used censoring scheme because its mechanism agrees with the design of experiments of many lifetime tests. The type I censored sample of the original complete ordered sample of failure times $x_1 \leq x_2 \leq \cdots \leq x_n$ on the right at a predetermined time t of the experiment consists of the first r failure times $x_1 \leq x_2 \leq \cdots \leq x_r$ that occur on or before t, and thus observed, where $r \leq n$. A convenient and simple explanation of the mathematical details of this type of censoring is available in [10]. The test accepts the null hypothesis that the distribution fits the data if the test statistic is larger than the critical value of the test and rejects otherwise, where the critical value is a cut-off point on the test distribution that is compared to the test statistics of the data to make the decision of the test. The area under the probability distribution of the test statistics to the right of the critical value is called the significance level of the test. That area gives the probability of rejecting the null hypothesis when it is true. On the other hand, the area to the right of the test statistic of the data is called the p-value of the test, where a high p-value suggests that the data are compatible and likely with the null hypothesis, whereas a low p-value suggests that the data do not support the null hypothesis and leads to its rejection. The p-value ranges from 0, corresponding to the extreme mismatch between the data and the distribution, to 1, corresponding to the perfect match between them.

In order to use the CVM or AD tests to measure the degree of agreement between given data and a theoretical distribution with CDF$F(x)$, we need to specify $F(x)$ by first estimating its parameters. Parameter estimation can be done using several ways available in literature. In this chapter we use the maximum likelihood estimators (MLEs) as they are most efficient, unbiased for large samples, and flexible and most importantly; can be applied to censored data as opposed to other types of estimators. On the other hand, they suffer from some drawbacks that each MLE problem must be specifically coded, in some cases, computing the MLE can only be done numerically, which might be sensitive to the initial values, and it depends on the complexity of the likelihood function of the treated distribution.

This chapter considers two of the most commonly used probability distributions in life studies and testing, namely, Weibull and lognormal distributions, whose basic properties are reviewed for the convenience of readers and are adapted from [11]. We denote by $f(x)$ and $F(x)$ the PDF and CDF, respectively.

- Weibull distribution denoted in this work by Weibull (α, β) where $\beta \in (0, \infty)$ is the scale parameter, and $\alpha \in (0, \infty)$ is the shape parameter, has PDF and CDF:

$$f(x) = \frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1} exp\left[-(\frac{x}{\beta})^{\alpha}\right], x \geq 0. \quad (6)$$

$$F(x) = 1 - exp\left[-(\frac{x}{\beta})^{\alpha}\right], x \geq 0 \quad (7)$$

The estimating equations of the MLEs from complete samples:

$$\frac{\sum_{i=1}^{n} x_i^{\hat{\alpha}_{com}} \log x_i}{\sum_{i=1}^{n} x_i^{\hat{\alpha}_{com}}} - \frac{1}{\hat{\alpha}_{com}} = \frac{1}{n}\sum_{i=1}^{n} \log x_i, \quad (8)$$

$$\hat{\beta}_{com} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{\alpha}_{com}}\right)^{1/\hat{\alpha}_{com}}. \tag{9}$$

The estimating equations of the MLEs from censored samples:

$$\begin{aligned} r = {} & (n-r)\left(t/\hat{\beta}_{cen}\right)^{\hat{\alpha}_{cen}} \log\left(t/\hat{\beta}_{cen}\right) \\ & -\sum_{i=1}^{r} \left[1-\left(x_i/\hat{\beta}_{cen}\right)^{\hat{\alpha}_{cen}}\right] \log\left(x_i/\hat{\beta}_{cen}\right)^{\hat{\alpha}_{cen}}, \end{aligned} \tag{10}$$

$$r = (n-r)\left(t/\hat{\beta}_{cen}\right)^{\hat{\alpha}_{cen}} + \sum_{i=1}^{r} \left(x_i/\hat{\beta}_{cen}\right)^{\hat{\alpha}_{cen}}. \tag{11}$$

- The lognormal distribution, Lognormal (θ, τ^2) where $\theta \in (-\infty, \infty)$ is the scale parameter, and $\tau \in (0, \infty)$ is the shape parameter, has PDF and CDF:

$$f(x) = \frac{1}{x\tau\sqrt{2\pi}} \quad exp\left[\frac{(\log x-\theta)^2]}{2\tau^2}\right], \ x \geq 0. \tag{12}$$

$$F(x) = \frac{1}{2} + \frac{1}{2} erf\left[\frac{\log x-\theta}{\sqrt{2}\tau}\right], \ x \geq 0. \tag{13}$$

The estimating equations of the MLEs from complete samples:

$$\hat{\theta}_{com} = \frac{1}{n}\sum_{i=1}^{n} \log x_i, \tag{14}$$

$$\hat{\tau}_{com} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} \left(\log x_i - \hat{\theta}_{com}\right)^2}. \tag{15}$$

The estimating equations of the MLEs from censored samples:

$$\hat{\theta}_{cen} = \frac{1}{r}\sum_{i=1}^{r} \log x_i + \frac{n-r}{r} \frac{\phi(z_{cen})}{1-\varphi(z_{cen})} \hat{\tau}_{cen}, \tag{16}$$

$$\hat{\tau}_{cen} = \sqrt{\frac{1}{r}\left(\log t - \hat{\theta}_{cen}\right)\sum_{i=1}^{r} \left(\hat{\theta}_{cen} - \log x_i\right) + \frac{1}{r}\sum_{i=1}^{r} \left(\log x_i - \hat{\theta}_{cen}\right)^2}, \tag{17}$$

where $z_{cen} = \frac{\log t - \hat{\mu}_{cen}}{\hat{\sigma}_{cen}}$, ϕ and φ are the PDF and CDF of the standard normal distribution, respectively.

For the lognormal distribution, the estimating equations (16) and (17) of the MLEs $\hat{\theta}_{cen}$ and $\hat{\tau}_{cen}$ from censored samples are both unimprovable implicit functions of $\hat{\theta}_{cen}$ and $\hat{\tau}_{cen}$. This makes it compulsory for us to solve them numerically, where the MLEs $\hat{\theta}_{com}$ and $\hat{\tau}_{com}$ can be used as starting estimates by just replacing *n* with *r* in equations (14) and (15), that is

$$\theta_0 = \frac{1}{r}\sum_{i=1}^{r} \log x_i, \tag{18}$$

$$\tau_0 = \sqrt{\frac{1}{r}\sum_{i=1}^{r} (\log x_i - \theta_0)^2}. \tag{19}$$

On the other hand, both estimating equations of the Weibull parameters from complete samples in equations (8) and (9), or censored samples in equations (10) and (11), are implicit and unimprovable, which must be numerically solved as well. Therefore, we need to have suitable starting estimates for $\hat{\alpha}_{cen}$ and $\hat{\beta}_{cen}$. The initial estimates can be obtained from the Weibull probability plot of the data as suggested by Nelson in [12], where $\hat{\alpha}_{cen}$ is the slope of the data in the log-log scale and $\hat{\beta}_{cen}$ is approximately the 63rd percentile. Although, the probability plots give good estimates or starting estimates to the parameters, they become useless when it comes to programing with fully automated steps. Dubey [13] studied the two-parameter Weibull distribution simple percentile estimators and provided, based on the simple percentiles, the following initial estimator for the Weibull distribution shape parameter.

$$\alpha_0 = \frac{Log[Log[1-p_k]] - Log[Log[1-p_j]]}{Log[x_k] - Log[x_j]}, \tag{20}$$

where x_k and x_j are any two different percentiles, and p_k and p_j are their percentile ranks. Dubey's initial estimate of the shape parameter α_0 in (18) is a programmable version of the slope proposed by Nelson. Dubey showed that the asymptotic variance of this estimator is minimum when $p_k = 0.17$ and $p_j = 0.97$. After finding the initial estimate α_0 of α from (18), the initial estimate of the scale parameter β can be obtained by substituting α_0 in (9) and replacing *n* with *r*

$$\beta_0 = \left(\frac{1}{r}\sum_{i=1}^{r} x_i^{\alpha_0}\right)^{1/\alpha_0}. \tag{21}$$

It can be checked for both distributions, in particular, and any distribution, in general, that the estimating equations from complete samples are special cases from the estimating equations from censored samples. Here, this can be verified by observing that when r is replaced by n, and after some algebraic manipulations, equations (10), (11), (16) and (17) turn into equations (8), (9), (14) and (15), respectively. This is also reflected on the CVM and AD statistics where the CVM statistics for censored sample in equation (4) immediately converts to equation (2) after replacing r with n and p with one, and with lengthier but straightforward calculations, the AD statistics for censored samples in equation (5) converts to equation (3). Therefore, the testing problem becomes more challenging if the data are censored.

Practitioners from engineering and applied sciences usually use MATLAB or Mathematica to run these tests. However, the existing built-in functions available on both programs are *only* applicable to tests for complete data. To be precise, the AD test of MATLAB gives correct results and can be immediately used for testing the goodness of fit of given complete data sets to theoretical probability distributions, including normal, exponential, extreme value, lognormal and Weibull distributions. However, its CVM test is not applicable to the practical situation of interest because it does not consider the modifications needed after estimating the underlying distribution's parameters. On the other hand, the commands of both tests on Mathematica are made to only handle complete data sets as well, but they still demand inserting the values of the parameters, which is not the case in real life applications, and both give completely erroneous analysis when used in this way.

In this chapter, the correct CVM and AD Mathematica codes for testing the appropriateness of Weibull and lognormal distributions to given type I censored data are provided. The two distributions are considered because of their intensive involvement in reliability and life studies, where in many cases, they are competing with each other, and it is desirable to select the best distribution to fit given data. Comparing the two distributions when modelling censored or complete data appears in several theoretical [14, 15, 16], and practical works considering experimental life data of energy and engineering systems, like estimating the life distribution of solar cells [1], commercial lithium-ion batteries [17, 18], and organic light emitting diodes [11]. [11] summarizes two previous works [19, 20] illustrating each distribution separately.

2. The pseudo code of the goodness-of-fit algorithms

This section presents the pseudo codes of CVM and AD algorithms for type I censored data, and the coming section provides the Mathematica codes of both tests for the Weibull and lognormal distributions.

Pseudocode is an informal way that is used to describe algorithms in preparation to write them using formal programing language syntax or software systems. It is used to outline and summarize program's flow without mentioning the programing details, where the purpose of writing a pseudocode is to design a program in mind in order to understand the program plan before implementation. In this work, the pseudo codes are presented in a very general way in order to make them suitable to any arbitrary life distribution that practitioners may wish to test. The algorithms for both tests depend on Monte Carlo simulation as a standing process, through which a massive number of censored samples from the underlying distribution, with parameters estimated from a given data, are simulated to produce the distribution of the test statistics. The distribution of the test statistics is in general unknown even for complete samples, and that of censored samples is even more complicated to be analytically derived. Consequently, the algorithms proposed in this chapter are time-consuming but safe and provide precise results. On the other hand, the existing asymptotic algorithms on MATLAB or Mathematica, which only handle complete and large samples, are faster but fail, of course, to test censored or small samples. The critical values of the two tests calculated by asymptotic algorithms are functions that solely depend on the size of the target sample and the distribution under consideration.

The pseudo codes of CVM and AD test are very similar, and only differ in the formula of the test statistics, namely, in steps 3) and 4) in the following bullet points that describe the pseudo codes of both tests.

Pseudo Code

1) Insert the following inputs: the data (data), censoring time (t), complete sample size (n), the number of simulated samples (M), the significance level of the test (sl).
2) Compute the MLEs of the parameters of the distribution under consideration from the given data in order to define the CDF of the distribution under consideration, say (F).
3) Compute the CVM or AD test statistics (TS) of the given data using the CDF (F).
4) Generate a random set of M CVM or AD test statistics, which forms the simulated test statistics distribution.
5) Determine the simulated critical value (CV) and the p-value (PV) of the test.
6) Display the outputs of the test as the parameters of the concerned distribution, the test statistics (TS), the critical value (CV), the p-value (PV), and the decision of the test as either "Accept” or "Reject” the hypothesis that the distribution fits the data.

3. The Mathematica codes of Cramér–von Mises and Anderson-Darling tests for Weibull and lognormal distributions

This section provides the Mathematica codes for testing goodness of fit of Weibull and lognormal distributions to a given type I censored sample data using CVM and AD tests. Each part of the codes consists of a group of related steps of the codes is titled by an instructional remark (*[1]*), (*[2]*),..,(*[6]*) corresponding to steps 1), 2),…, 6) of the pseudo code. The text inserted between two asterisks (* Text *) is nonexecutable within the body of a program while running the codes in

Mathematica and will be automatically colored in light blue or gray when typed in the program depending in the Mathematica version used. This is equivalent to the percent sign (%) in MATLAB appearing in light green. Like MATLAB, semicolons (;) are placed at the end of some commands to suppress their output, and some other commands don't have semicolons as we need to see their outputs. As clear from the first bullet point in the pseudo code, the users must insert the censored data set data) to be tested. In Mathematica, this set is defined using set brackets {}, and the values inside the set must be separated by commas. That is the data must look like this data={value(1), value(2),…,value(r)}. In addition to data, the user must insert four parameters

- The censoring time (t), a positive real number that must be more than the largest observed value in the data, say value(r).
- The complete sample size (n), a positive integer that is equal to the number of values of the complete sample, where the complete sample consists of the observed failures (data) and the unobserved failures due to censoring.
- The number of simulated samples (M), a large positive integer that is equal to the number of runs or loops used to generate a big set of test statistics. We recommend the users to set M to be at least 5000 to achieve precise results but not more than 10000 to have reasonable running times. Here we use M=10000 to achieve high accuracy.
- The significance level (sl), a small percentage that needs to be determined before conducting the test, and it is the probability of rejecting the hypothesis that the data follow the underlying distribution (null hypothesis) when it is true. The most commonly used significant level is 0.05.

The first part of the codes is indexed by (*[1]*) and consists of the five inputs highlighted in red to notify the user to insert these quantities. In the second part, which is indexed by (*[2]*), the codes compute the MLEs of the parameters by solving the estimating equations (10) and (11) for Weibull, or (16) and (17) for lognormal presented in Section 1. The estimating equations are solved using the built-in function “FindRoot”, which searches a solution using Newton–Raphson method and requires initial estimates of the parameters. The initial estimates are obtained from (18) and (19) for lognormal, and (20) and (21) for Weibull. The test statistic (TS) of the given data is computed in the third part (*[3]*) using equation (4) for CVM and (5) for AD. The most important part of the code is (*[4]*), in which the code simulates a big number of random test statistics generated through Monte Carlo simulation. This task can be implemented in several ways, but we opt to use “Block”. Block is a closed box that localizes symbols by separating their values from values that might assume outside the block. The idea is very simple, as we just need to generate big number of random censored samples and then compute their corresponding test statistics. The blocks can be thought of as “For Loop”, but they have the significance that their output(s) are functions of their input(s) making them easy to be understood and implemented. After defining the block in part (*[4]*), in part (*[5]*), the code runs the block for a big number of times “M” that has been already inserted in part (*[1]*), to create the simulated test statistics distribution, from which the simulated critical values (CV) is determined. The CV is usually chosen as the 0.95^{th} percentile corresponding to the significance level (sl) of 0.05 already inserted in part (*[1]*). Then the simulated PV is the percentage of the simulated test statistics that are more than TS. The reader may notice that in part (*[5]*), the built-in function “Quiet” is used to mute Mathematica to prevent showing the messages on the precision of the calculations of some few specimens generated. Also, the generated values of test statistics are provided with the function Re[] inside the critical value command to only consider the real parts of few generated test statistics with nonzero imaginary part. Part (*[5]*) is also supported with the command b=Complement[b,{Indeterminate}]. This command excludes any Indeterminate Value of the test statistics that may happen due to many reasons such as the simulated censored sample size is 0 or an error that may occur while solving the MLE estimating equations. In the final part (*[6]*), we just display the quantities that we need to see, which are the estimated parameters computed in part (*[2]*), and the CV and PV determined in part (*[5]*), in addition to the decision of the test as “accept” when PV>sl or “reject” otherwise. In the AD test code, a negligible number of 10^{-100} is added inside the $\mathrm{Log}[1-p+10^{-100}]$ in the test statistics of the block because the size of some specimens might be equal to the complete sample size and thus $p=1$ making $\mathrm{Log}[1-p]$ undefined otherwise.

In the following, the CVM and AD codes are presented for the two distributions. The users need to type them in a Mathematica sheet for implementation.

Code I. The Cramér–von Mises test Mathematica code for Weibull distribution

```
(*[1] The Inputs of the Code

data: the observed data, t: the time of censoring, n: the complete sample size, M: 
number of sets of specimens, sl: the significance level*)

data= {}

t=

n=

M=

Sl=
```

```
(*[2] computing the MLEs of α and β of the given data*)

r = Length[data];
```

$$\alpha0=\frac{\mathrm{Log}[\mathrm{Log}[1-0.97]]-\mathrm{Log}[\mathrm{Log}[1-0.17]]}{\mathrm{Log}[\mathrm{Quantile}[\mathrm{data},0.97]]-\mathrm{Log}[\mathrm{Quantile}[\mathrm{data},0.17]]};$$

$$\beta0=(\frac{1}{\mathrm{r}}*\mathrm{Total}[\mathrm{data}^{\alpha0}])^{1/\alpha0};$$

$$\mathrm{A}[\mathrm{c\_},\mathrm{d\_}]=-(\mathrm{n}-\mathrm{r})(\mathrm{t}/\mathrm{c})^{\mathrm{d}}*\mathrm{Log}[\mathrm{t}/\mathrm{c}]+\mathrm{r}+\mathrm{Total}[\mathrm{Log}[(\mathrm{data}/\mathrm{c})^{\mathrm{d}}]]-\mathrm{Total}[(\mathrm{data}/\mathrm{c})^{\mathrm{d}}*\mathrm{Log}[(\mathrm{data}/\mathrm{c})^{\mathrm{d}}]];$$

$$\mathrm{B}[\mathrm{c\_},\mathrm{d\_}]=(\mathrm{n}-\mathrm{r})*(\mathrm{t}/\mathrm{c})^{\mathrm{d}}+\mathrm{Total}[(\mathrm{data}/\mathrm{c})^{\mathrm{d}}]-\mathrm{r};$$

```
α = FindRoot[{A[c, d] == 0, B[c, d] == 0}, {{d, α0}, {c, β0}}][[1]][[2]];

β = FindRoot[{A[c, d] == 0, B[c, d] == 0}, {{d, α0}, {c, β0}}][[2]][[2]];
```

```
(*[3] The Test Statistics of the Observed Data*)

F[x_] = 1 − Exp[−(x/β)α];

dataSORT = Sort[data] ;

z = F[dataSORT];

p = r/n;

TS = Sum[i=1..r] (z[[i]] − (2i − 1)/(2 ∗ n))^2 − (r ∗ (4 ∗ r^2 − 1))/(12 ∗ n^2) + n ∗ p ∗ ((r^2)/(n^2) − p ∗ (r/n) + 1/3 ∗ p^2);
```

```
(*[4] The Simulated Test Statistics Distribution *)

CVMtestW[α_, β_, n_, t_] ≔ Block[{UNIFdata, COMdata, COMdataSORT, r, CENdata, αcen, βcen, F1, z, p, CVM, A, B},

UNIFdata = RandomReal[{0,1}, n];

COMdata = β ∗ (−1 ∗ Log[1 − UNIFdata])^(1/α);

COMdataSORT = Sort[COMdata];

r = Count[COMdataSORT, u_ /; u ≤ t];

CENdata = Table[COMdataSORT[[i]], {i, 1, r}];

A[c_, d_] = −(n − r)(t/c)^d ∗ Log[t/c] + r + Total[Log[(CENdata/c)^d]] − Total[(CENdata/c)^d ∗ Log[(CENdata/c)^d]] ;

B[c_, d_] = (n − r) ∗ (t/c)^d + Total[(CENdata/c)^d] − r;

αcen = FindRoot[{A[c, d] == 0, B[c, d] == 0}, {{d, α}, {c, β}}][[1]][[2]];

βcen = FindRoot[{A[c, d] == 0, B[c, d] == 0}, {{d, α}, {c, β}}][[2]][[2]];

F1[x_] = 1 − Exp[−(x/βcen)^αcen];

z = F1[CENdata];

p = r/n ;

CVM = Sum[i=1..r] (z[[i]] − (2i − 1)/(2 ∗ n))^2 − (r ∗ (4 ∗ r^2 − 1))/(12 ∗ n^2) + n ∗ p ∗ ((r^2)/(n^2) − p ∗ (r/n) + 1/3 ∗ p^2);

{CVM}]
```

```
(*[5] The Simulated Critical and p Values*)

a = Quiet[Table[CVMtestW[α, β, n, t], M]];

 b = Table[a[[i]][[1]], {i, 1, M}];
```

```
b = Complement[b, {Indeterminate}];

CV = Quantile[Re[b],1 − sl];

PV = N[Length[Select[b, # > TS &]]/M];
```

```
(*[6] The Outputs of the Code*)

Print["The CVM Test Results for Weibull Distribution"]
Print["The MLE of α = "]
α
Print["The MLE of β = "]
β
Print["The test statistics of the observed data = " ]
TS
Print["The simulated critical value of the test = " ]
CV
Print["The simulated p − value = " ]
PV
If[PV > sl, Print["Decision: ′Accept′"], Print["Decision: ′Reject′"]]
```

Code II. The Cramér–von Mises test Mathematica code for the lognormal distribution

```
(*[1] The Inputs of the Code

data: the observed data, t: the time of censoring, n: the complete sample size, M:
number of sets of specimens, sl: the significance level*)data= {}

t=

n=

M=

Sl=
```

```
(*[2] computing the MLEs of α and β of the given data*)

r = Length[data];
```

$$\theta0 = \mathrm{N}\left[\frac{\mathrm{Total}\left[\mathrm{Log}[\mathrm{data}]\right]}{\mathrm{r}}\right];$$

$$\tau0 = \left(\frac{\mathrm{Total}[(\mathrm{Log}[\mathrm{data}] - \theta0)^2]}{\mathrm{r}}\right)^{0.5};$$

$$\mathrm{A}\left[\mathrm{m\_,s\_}\right] = (\mathrm{Log}[\mathrm{t}] - \mathrm{m}) * \left(\mathrm{m} - \frac{\mathrm{Total}\left[\mathrm{Log}[\mathrm{data}]\right]}{\mathrm{r}}\right) + \frac{\mathrm{Total}[(\mathrm{Log}[\mathrm{data}] - \mathrm{m})^2]}{\mathrm{r}} - \mathrm{s}^2;$$

$$\mathrm{B}\left[\mathrm{m\_,s\_}\right] = \frac{\mathrm{Total}\left[\mathrm{Log}[\mathrm{data}]\right]}{\mathrm{r}} + \frac{\mathrm{n} - \mathrm{r}}{\mathrm{r}} * \mathrm{s} * \frac{\mathrm{PDF}\left[\mathrm{NormalDistribution}[0,1], \frac{\mathrm{Log}[\mathrm{t}] - \mathrm{m}}{\mathrm{s}}\right]}{1 - \mathrm{CDF}\left[\mathrm{NormalDistribution}[0,1], \frac{\mathrm{Log}[\mathrm{t}] - \mathrm{m}}{\mathrm{s}}\right]} - \mathrm{m};$$

```
τ = FindRoot[{A[m, s] == 0, B[m, s] == 0}, {{m, θ0}, {s, τ0}}][[2]][[2]];
```

θ = FindRoot[{A[m, s] == 0, B[m, s] == 0}, {{m, θ0}, {s, τ0}}][[1]][[2]];

`(*[3] The Test Statistics of the Observed Data*)`

$$F[x\_] = \frac{1}{2}(1 + \text{Erf}[\frac{-\theta + \text{Log}[x]}{\sqrt{2}\tau}]);$$

dataSORT = Sort[data] ;

z = F[dataSORT];

$p = r/n;$

$$TS = \sum_{i=1}^{r}(z[[i]] - (2i - 1)/(2 * n))\wedge 2 - \frac{r * (4 * r\wedge 2 - 1)}{12 * n\wedge 2} + n * p * (\frac{r\wedge 2}{n\wedge 2} - p * (\frac{r}{n}) + \frac{1}{3} * p\wedge 2);$$

`(*[4] The Simulated Test Statistics Distribution *)`

CVMtestL[θ_, τ_, n_, t_] := Block[{UNIFdata, COMdata, COMdataSORT, r, CENdata, θcen, τcen, F1, z, p, CVM},

UNIFdata = RandomReal[{0,1}, n];

$$\text{COMdata} = \text{Exp}\left[\sqrt{2} * \tau * \text{InverseErf}[2 * \text{UNIFdata} - 1] + \theta\right];$$

COMdataSORT = Sort[COMdata];

r = Count[COMdataSORT, u_ /; u ≤ t];

CENdata = Table[COMdataSORT[[i]], {i, 1, r}];

$$A[m\_, s\_] = (\text{Log}[t] - m) * \left(m - \frac{\text{Total}[\text{Log}[\text{CENdata}]]}{r}\right) + \frac{\text{Total}[(\text{Log}[\text{CENdata}] - m)^2]}{r} - s^2;$$

$$B[m\_, s\_] = \frac{\text{Total}[\text{Log}[\text{CENdata}]]}{r} + \frac{n - r}{r} * s * \frac{\text{PDF}\left[\text{NormalDistribution}[0,1], \frac{\text{Log}[t] - m}{s}\right]}{1 - \text{CDF}\left[\text{NormalDistribution}[0,1], \frac{\text{Log}[t] - m}{s}\right]} - m;$$

θcen = FindRoot[{A[m, s] == 0, B[m, s] == 0}, {{m, θ}, {s, τ}}][[1]][[2]];

τcen = FindRoot[{A[m, s] == 0, B[m, s] == 0}, {{m, θ}, {s, τ}}][[2]][[2]];

$$F1[x\_] = \frac{1}{2}\left(1 + \text{Erf}\left[\frac{-\theta\text{cen} + \text{Log}[x]}{\sqrt{2}\tau\text{cen}}\right]\right);$$

z = F1[CENdata];

p = r/n ;

$$\text{CVMdata} = \sum_{i=1}^{r}(z[[i]] - (2i - 1)/(2 * n))\wedge 2 - \frac{r * (4 * r\wedge 2 - 1)}{12 * n\wedge 2} + n * p * (\frac{r\wedge 2}{n\wedge 2} - p * (\frac{r}{n}) + \frac{1}{3} * p\wedge 2);$$

{CVM}]

`(*[5] The Simulated Critical and p Values*)`

```
a = Quiet[Table[CVMtestL[θ, τ, n, t], M]];

b = Table[a[[i]][[1]], {i, 1, M}];

b = Complement[b, {Indeterminate}];

CV = Quantile[Re[b],1 − sl];

PV = N[Length[Select[b, # > TS &]]/M];
```

```
(*[6] The Outputs of the Code*)

Print["The CVM Test Results for Lognormal Distribution"]
Print["The MLE of θ = "]
θ
Print["The MLE of τ = "]
τ
Print["The test statistics of the observed data = " ]
TS
Print["The simulated critical value of the test = " ]
CV
Print["The simulated p − value = " ]
PV
If[PV > sl, Print["Decision: ′Accept′"], Print["Decision: ′Reject′"]]
```

Code III. The Anderson-Darling test Mathematica code for Weibull distribution

```
(*[1] The Inputs of the Code

data: the observed data, t: the time of censoring, n: the complete sample size, M:
number of sets of specimens, sl: the significance level*)

data= {}

t=

n=

M=

Sl=
```

The MLEs of α and β

```
(*[2] computing the MLEs of α and β of the given data*)

r = Length[data];
```

$$\alpha0 = \frac{\text{Log[Log[1 − 0.97]] − Log[Log[1 − 0.17]]}}{\text{Log[Quantile[data, 0.97]] − Log[Quantile[data, 0.17]]}};$$

$$\beta0 = (\frac{1}{r} * \text{Total[data}^{\alpha0}\text{]})^{1/\alpha0};$$

$$\text{A[c\_, d\_]} = -(n - r)(t/c)^{d} * \text{Log}[t/c] + r + \text{Total[Log[(data/c)}^{d}]] - \text{Total[(data/c)}^{d} * \text{Log[(data/c)}^{d}]];$$

```
B[c_, d_] = (n − r) ∗ (t/c)^d + Total[(data/c)^d] − r;

α = FindRoot[{A[c, d] == 0, B[c, d] == 0}, {{d, α0}, {c, β0}}][[1]][[2]];

β = FindRoot[{A[c, d] == 0, B[c, d] == 0}, {{d, α0}, {c, β0}}][[2]][[2]];
```

```
(*[3] The Test Statistics of the Observed Data*)

F[x_] = 1 − Exp[−(x/β)α];

dataSORT = Sort[data] ;

z = F[dataSORT];

p = r/n;
```

$$TS = \sum_{i=1}^{r} (2i - 1) * (\mathrm{Log}[1 - z[[i]]] - \mathrm{Log}[z[[i]]]) /n - 2 * \sum_{i=1}^{r} \mathrm{Log}[1 - z[[i]]] + n * (\frac{2 * r}{n} - (\frac{r}{n})\hat{}2 - 1) * \mathrm{Log}[1 - p] + \frac{r\hat{}2}{n} * \mathrm{Log}[p] - p * n;$$

```
(*[4] The Simulated Test Statistics Distribution *)

ADtestW[α_, β_, n_, t_] := Block[{UNIFdata, COMdata, COMdataSORT, r, CENdata, αcen, βcen, F1, z, p, AD, A, B},

UNIFdata = RandomReal[{0,1}, n];

COMdata = β ∗ (−1 ∗ Log[1 − UNIFdata])^(1/α);

COMdataSORT = Sort[COMdata];

r = Count[COMdataSORT, u_ /; u ≤ t];

CENdata = Table[COMdataSORT[[i]], {i, 1, r}];

A[c_, d_] = −(n − r)(t/c)^d ∗ Log[t/c] + r + Total[Log[(CENdata/c)^d]] − Total[(CENdata/c)^d ∗ Log[(CENdata/c)^d]];

B[c_, d_] = (n − r) ∗ (t/c)^d + Total[(CENdata/c)^d] − r;

αcen = FindRoot[{A[c, d] == 0, B[c, d] == 0}, {{d, α}, {c, β}}][[1]][[2]];

βcen = FindRoot[{A[c, d] == 0, B[c, d] == 0}, {{d, α}, {c, β}}][[2]][[2]];

F1[x_] = 1 − Exp[−(x/βcen)^αcen];

z = F1[CENdata];

p = r/n ;
```

$$AD = \sum_{i=1}^{r} (2i - 1) * \left(\mathrm{Log}\left[1 - z[[i]]\right] - \mathrm{Log}\left[z[[i]]\right]\right) /n - 2 * \sum_{i=1}^{r} \mathrm{Log}\left[1 - z[[i]]\right] + n * \left(\frac{2 * r}{n} - \left(\frac{r}{n}\right)^2 - 1\right) * \mathrm{Log}[1 - p + 10^{-100}] + \frac{r^2}{n} * \mathrm{Log}[p] - p * n;$$

```
{AD}]
```

```
(*[5] The Simulated Critical and p Values*)

a = Quiet[Table[ADtestW[α, β, n, t], M]];

 b = Table[a[[i]][[1]], {i, 1, M}];

b = Complement[b, {Indeterminate}];

CV = Quantile[Re[b],1 − sl];

PV = N[Length[Select[b, # > TS &]]/M];
```

```
(*[6] The Outputs of the Code*)

Print["The AD Test Results for Weibull Distribution"]
Print["The MLE of α = "]
α
Print["The MLE of β = "]
β
Print["The test statistics of the observed data = " ]
TS
Print["The simulated critical value of the test = " ]
CV
Print["The simulated p − value = " ]
PV
If[PV > sl, Print["Decision: ′Accept′"], Print["Decision: ′Reject′"]]
```

Code IV. The Anderson-Darling test Mathematica code for the lognormal distribution

```
(*[1] The Inputs of the Code

data: the observed data, t: the time of censoring, n: the complete sample size, M:
number of sets of specimens, sl: the significance level*)

data= {}

t=

n=

M=

Sl=
```

```
(*[2] computing the MLEs of α and β of the given data*)

r = Length[data];
```

$$\theta 0 = \mathrm{N}\left[\frac{\mathrm{Total}[\mathrm{Log}[\mathrm{data}]]}{\mathrm{r}}\right];$$

$$\tau 0 = \left(\frac{\mathrm{Total}[(\mathrm{Log}[\mathrm{data}] - \theta 0)^{2}]}{\mathrm{r}}\right)^{0.5};$$

$$A[m\_,s\_] = (\mathrm{Log}[t] - m) * \left(m - \frac{\mathrm{Total}[\mathrm{Log}[data]]}{r}\right) + \frac{\mathrm{Total}[(\mathrm{Log}[data] - m)^2]}{r} - s^2;$$

$$B[m\_,s\_] = \frac{\mathrm{Total}[\mathrm{Log}[data]]}{r} + \frac{n-r}{r} * s * \frac{\mathrm{PDF}\left[\mathrm{NormalDistribution}[0,1], \frac{\mathrm{Log}[t]-m}{s}\right]}{1 - \mathrm{CDF}\left[\mathrm{NormalDistribution}[0,1], \frac{\mathrm{Log}[t]-m}{s}\right]} - m;$$

τ = FindRoot[{A[m, s] == 0, B[m, s] == 0}, {{m, θ0}, {s, τ0}}][[2]][[2]];

θ = FindRoot[{A[m, s] == 0, B[m, s] == 0}, {{m, θ0}, {s, τ0}}][[1]][[2]];

`(*[3] The Test Statistics of the Observed Data*)`

$$F[x\_] = \frac{1}{2}(1 + \mathrm{Erf}[\frac{-\theta + \mathrm{Log}[x]}{\sqrt{2}\tau}]);$$

dataSORT = Sort[data] ;

z = F[dataSORT];

p = r/n;

$$TS = \sum_{i=1}^{r}(2i-1) * (\mathrm{Log}[1 - z[[i]]] - \mathrm{Log}[z[[i]]])\,/n - 2 * \sum_{i=1}^{r}\mathrm{Log}[1 - z[[i]]] + n * (\frac{2*r}{n} - (\frac{r}{n})\text{^}2 - 1) * \mathrm{Log}[1-p] + \frac{r\text{^}2}{n} * \mathrm{Log}[p] - p * n;$$

`(*[4] The Simulated Test Statistics Distribution *)`

ADtestL[θ_, τ_, n_, t_] := Block[{UNIFdata, COMdata, COMdataSORT, r, CENdata, θcen, τcen, F1, z, p, AD},

UNIFdata = RandomReal[0,1, n];

$$COMdata = \mathrm{Exp}[\sqrt{2} * \tau * \mathrm{InverseErf}[2 * UNIFdata - 1] + \theta];$$

COMdataSORT = Sort[COMdata];

r = Count[COMdataSORT, u_ /; u ≤ t];

CENdata = Table[COMdataSORT[[i]], {i, 1, r}];

$$A[m\_,s\_] = (\mathrm{Log}[t] - m) * \left(m - \frac{\mathrm{Total}[\mathrm{Log}[CENdata]]}{r}\right) + \frac{\mathrm{Total}[(\mathrm{Log}[CENdata] - m)^2]}{r} - s^2;$$

$$B[m\_,s\_] = \frac{\mathrm{Total}[\mathrm{Log}[CENdata]]}{r} + \frac{n-r}{r} * s * \frac{\mathrm{PDF}\left[\mathrm{NormalDistribution}[0,1], \frac{\mathrm{Log}[t]-m}{s}\right]}{1 - \mathrm{CDF}\left[\mathrm{NormalDistribution}[0,1], \frac{\mathrm{Log}[t]-m}{s}\right]} - m;$$

θcen = FindRoot[{A[m, s] == 0, B[m, s] == 0}, {{m, θ}, {s, τ}}][[1]][[2]];

τcen = FindRoot[{A[m, s] == 0, B[m, s] == 0}, {{m, θ}, {s, τ}}][[2]][[2]];

$$F1[x\_] = \frac{1}{2}\left(1 + \mathrm{Erf}\left[\frac{-\theta cen + \mathrm{Log}[x]}{\sqrt{2}\tau cen}\right]\right);$$

z = F1[CENdata];

```
p = r/n;
```

$$\mathrm{AD} = \sum_{i=1}^{r}(2i-1)*\left(\mathrm{Log}\left[1-z[[i]]\right]-\mathrm{Log}\left[z[[i]]\right]\right)/n - 2*\sum_{i=1}^{r}\mathrm{Log}\left[1-z[[i]]\right] + n*\left(\frac{2*r}{n}-\left(\frac{r}{n}\right)^{2}-1\right) * \mathrm{Log}[1-p+10^{-100}] + \frac{r^{2}}{n}*\mathrm{Log}[p] - p*n;$$

```
{AD}]
```

```
(*[5] The Simulated Critical and p Values*)

α = Quiet[Table[ADtestL[θ, τ, n, t], M]];

 b = Table[α[[i]][[1]], {i, 1, M}];

b = Complement[b, {Indeterminate}];

CV = Quantile[Re[b],1 − sl];

PV = N[Length[Select[b, # > TS &]]/M];
```

```
(*[6] The Outputs of the Code*)

Print["The AD Test Results for Lognormal Distribution"]
Print["The MLE of θ = "]
θ
Print["The MLE of τ = "]
τ
Print["The test statistics of the observed data = " ]
TS
Print["The simulated critical value of the test = " ]
CV
Print["The simulated p − value = " ]
PV
If[PV > sl, Print["Decision: ′Accept′"], Print["Decision: ′Reject′"]]
```

4. Illustrative Examples

In this section, the codes presented in Section 3 are applied to several examples extracted from literature. The examples illustrate the goodness of fit of real data sets to Weibull and lognormal distributions. The data sets consist of lives of electrical devises that are used in energy systems.

Example 4.1: (Solar Cells) Our first example is on the life of concentrated solar cells with experimental life data adapted from [1]. In [1], a sample of 45 commercial concentrators lattice-matched GaInP/GaInAs/Ge cells were equally segregated and exposed to three temperature levels; T_1: 164°C (437K), T_2: 126°C (399K), and T_3: 119°C (392K). The time a cell was able to endure that temperature was recorded, reflecting the cell lifetime. The operation was replicated by injecting current in the darkness, which is equivalent to the photogenerated current by cells subjected to the actual field irradiance of 820X. At the lowest temperature intensity T_3, the test took a long time to cause failures and the experiment was terminated after the failure of the 9th cell. This segment of cells is considered censored as some are still working, and thus their exact lifetimes are unknown, whereas the two higher temperature levels T_1 and T_2 resulted in the failure of all cells. The failure times of the cells are not explicitly reported in [1], where the authors present them through Weibull probability plots to graphically demonstrate the compliance of the data to this distribution. However, graphical methods are considered subjective and analytical methods are needed in order to achieve a comprehensive analysis. Here we focus on the third stress T_3 as the resulting data is censored. The failure times are extracted from Figure 7 of [1] and listed in Table 1.

Table 1: The failure times, in hours, of the solar cells of Example 4.1 under stress level T_3

1300	1500	1850	1950	2490
2490	3280	3510	3515	

Since the maximum failure time is 3515 hours, it might be reasonable to assume that the test was terminated after 3600 hours. Now let us test the compliance of the data to the two models considered in this work using the codes of CVM and AD tests for censored data introduced in Sections 3.

In order to run the codes, users must insert the data (data), censoring time (t), the complete sample size (n), the number of simulations (M) and the significance level (sl). These inputs must look like this

```
data={1300, 1500, 1850, 1950, 2490, 2490, 3280, 3510, 3515};
t=3600;
```

```
n=15;

M=10000;

sl=0.05;
```

After running the two codes, the outputs will be as follows:

Table 2: The CVM test results for Weibull and lognormal distributions of the solar cells life data provided in Table 1

Code I	**Code II**
The Cramer-von Mises Test Results for Weibull Distribution The MLE of α= 2.60231 The MLE of β= 3745.84 The test statistics of the observed data= 0.025052 The simulated critical value of the test= 0. 0697111 The simulated p-value= 0. 4482 Decision: 'Accept'	The Cramer-von Mises Test Results for Lognormal Distribution The MLE of θ= 8.06732 The MLE of τ= 0.517784 The test statistics of the observed data= 0.0245141 The simulated critical value of the test= 0.0674172 The simulated p-value= 0.4511 Decision: 'Accept'

Table 3: The AD test results for Weibull and lognormal distributions of the solar cells life data provided in Table 1

Code III	**Code IV**

```
The Anderson-Darling
Test Results for Weibull Distributio
n

The MLE of α=
2.60231

The MLE of β=
3745.84

The test statistics of the observed
data=
0.147656

The simulated critical value of the
test=
0. 420839

The simulated p-value=
0. 5293

Decision: 'Accept'
```

```
The Anderson-Darling
Test Results for Lognormal Distributio
n

The MLE of θ=
8.06732

The MLE of τ=
0.517784

The test statistics of the observed
data=
0.136188

The simulated critical value of the
test=
0. 382118

The simulated p-value=
0. 5579

Decision: 'Accept'
```

It can be seen from the outputs of both tests that the two distributions well fit the experimental data but with slight preference to the lognormal distribution as it achieves higher p-values. The authors of [1] adopted, based on a graphical approach, the Weibull distribution as the best fitting model for their data, whereas we find here from the calculations above that the lognormal is at least better than Weibull. Practitioners must be careful when using the codes or built-in functions available in the market when testing censored samples. For example, the MATLAB function of the CVM (cmtest) for Weibull distribution demands inserting the estimates of its parameters, the scale β, and shape α parameters. From the above output, β=3745.84 and α=2.60231. When implementing the CVM test of MATLAB as follows

```
>> data={1300, 1500, 1850 , 1950, 2490, 2490, 3280, 3510, 3515};
```

```
>> Beta=3745.84; Alpha=2.60231;
>>[H,PV,TS,CV]=cmtest(data′,′alpha′,0.05,′CDF′,[data′,wblcdf(data′,Beta,Alpha
)])
```

the result will be as follows

```
H=1            P=0.0420            TS=0.4811           CV=0.4534
```

H is 1 means that the test rejects the null hypothesis that the data follows the distribution under consideration, which is Weibull in this case, and 0 otherwise. PV, TS and CV are the p-value, the test statistic and the critical value of the test, respectively. This example shows the limitations of this command on MATLAB as it strongly rejects the data to follow the Weibull distribution though this distribution properly fits the data as clear from the probability plot in Fig. 1. Probability plots consist of a straight line that represents the theoretical distribution and points representing the experimental data. The data points fall on or close to the straight line of the Weibull distribution indicating its appropriateness to the data [21]. On the other hand, if one tries to find the maximum likelihood estimates of the given data using the built-in command “wblfit” of MATLAB, the result will be β=2716.8 and α=3.4000, which are different but comparable to the correct estimates mentioned above, β=3745.84 and α=2.60231. The reason for having different results is that the wblfit considers the data as a complete sample and not censored. Now, when we run, using β=2716.8 and α=3.4000, the cmtest of MATLAB as follows

```
>> data={1300, 1500, 1850 , 1950, 2490, 2490, 3280, 3510, 3515};
>> Beta=2716.8; Alpha=3.4000;
>>[H,P,TS,CV]=cmtest(data′,′alpha′,0.05,′CDF′,[data′,wblcdf(data′,Beta,Alpha)
])
```

we get

```
H=0            P=0.8400            TS=0.0579           CV=0.4534
```

The new p-value is now p=0.8400 (strongly accept), which is too optimistic, whereas the previous p-value is 0.0420 (strongly reject). This big difference between the two decisions demonstrates the shortcomings of the command and the mistake that practitioners would commit.

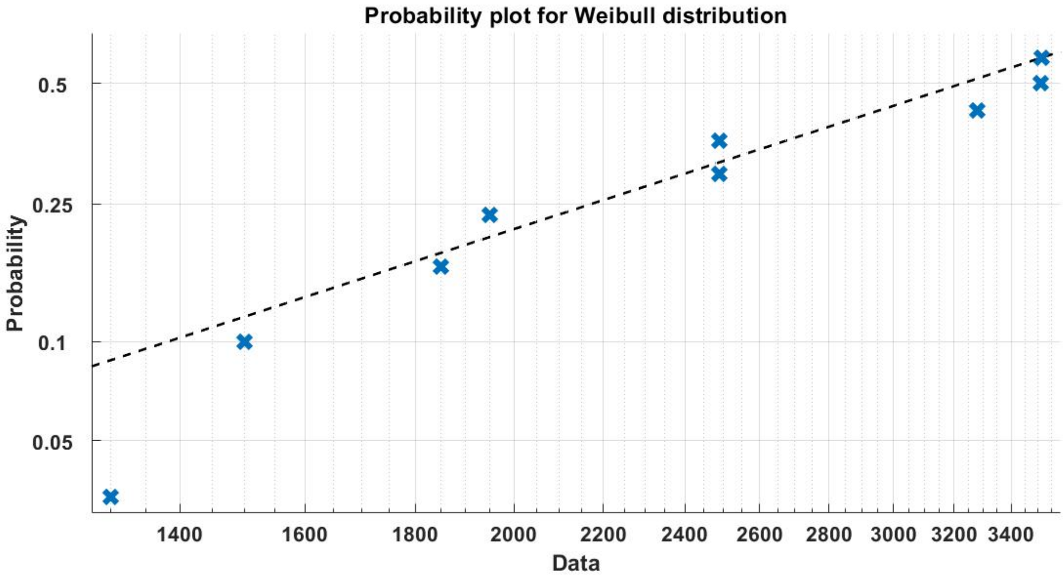

Fig. 1. The Weibull probability plot of the data in Table 1.

The same wrong result can appear on Mathematica when using the built-in command CramerVonMisesTest[data, WeibullDistribution[2.60231, 3745.84]] producing a p-value of 0.0444 declaring the rejection of the Weibull distribution, and this is completely wrong. Fig. 2 depicts the lognormal probability plot of the given data. The plot of lognormal distribution is very similar to that of Weibull, but the former has a clear better fit on the left tail of the data.

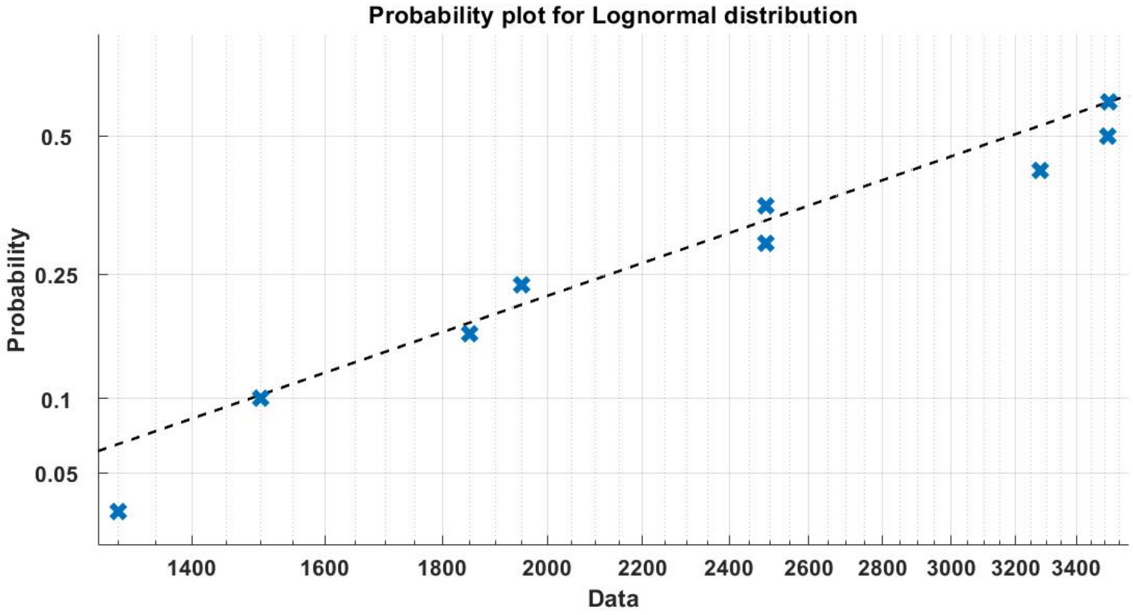

Fig. 2. The lognormal probability plot of the data in Table 1.

The two figures show that the two distributions are suitable models to the given data with preference to lognormal distribution in complete agreement with the goodness-of-fit results presented in Tables 2 and 3.

Example 4.2: (Lithium-Ion Batteries) In [17], the authors consider the problem of choosing the best life model, Weibull or lognormal, of commercial lithium-ion batteries based on experimental data published in [18]. After using four goodness-of-fit tests and the efficiency of censored sample indicator [22], they conclude, based on the usual asymptotic formula of CVM, that the lognormal outperforms Weibull. The experimental data consists of failure times, measured in cycles, of 20 batteries out of 24 in total after exposing them to high stress, where the batteries are cycled at 25 ^{o}C with an Arbin BT2000. Each of them is charged in a constant current constant voltage mode at 1C (4.4 A) constant current up to 4.35V until the current decreases below C/40. Finally, the batteries are discharged at 10C (44 A) constant current until the terminal voltage dropped to 3 V. The test is terminated after 593 cycles of continuous charge and discharge, with only four survivors out of 24 batteries, and thus t=593 can be considered as the point of censoring. Table 4 depicts the cycles by which the 20 failures happen.

Table 4: The failure cycles of the 20 batteries of Example 4.2

255	301	326	338	340
341	379	408	409	430
449	475	497	509	515
518	537	541	541	560

Figs 3 and 4 depicts the probability plots of the data shown in Table 4 using the two models considered in this chapter.

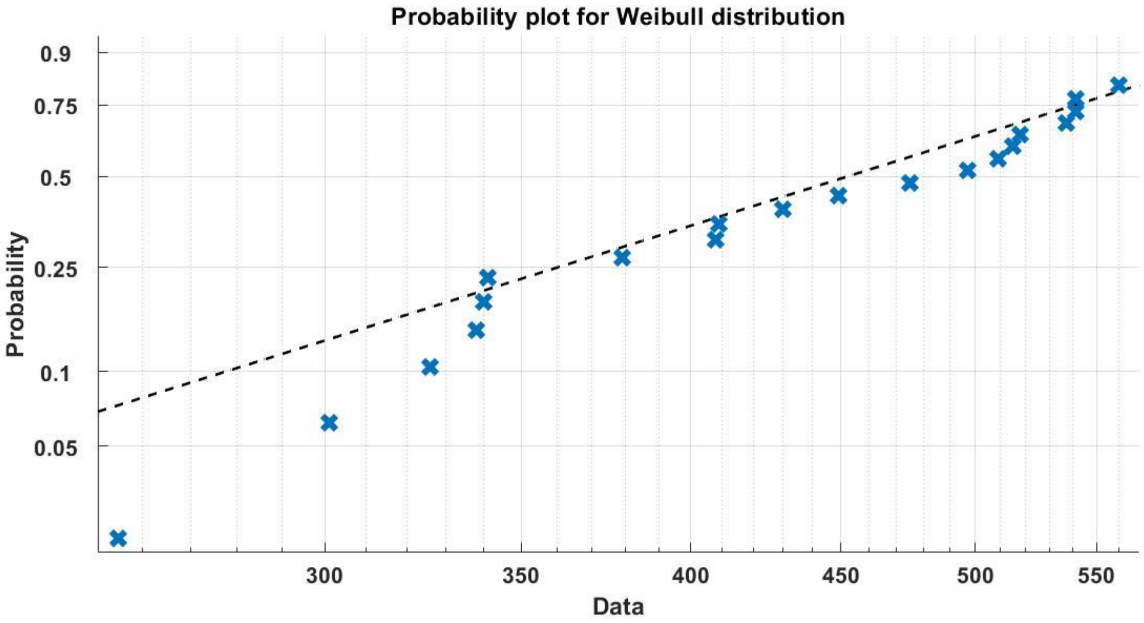

Fig. 3. The Weibull probability plot of the data in Table 4.

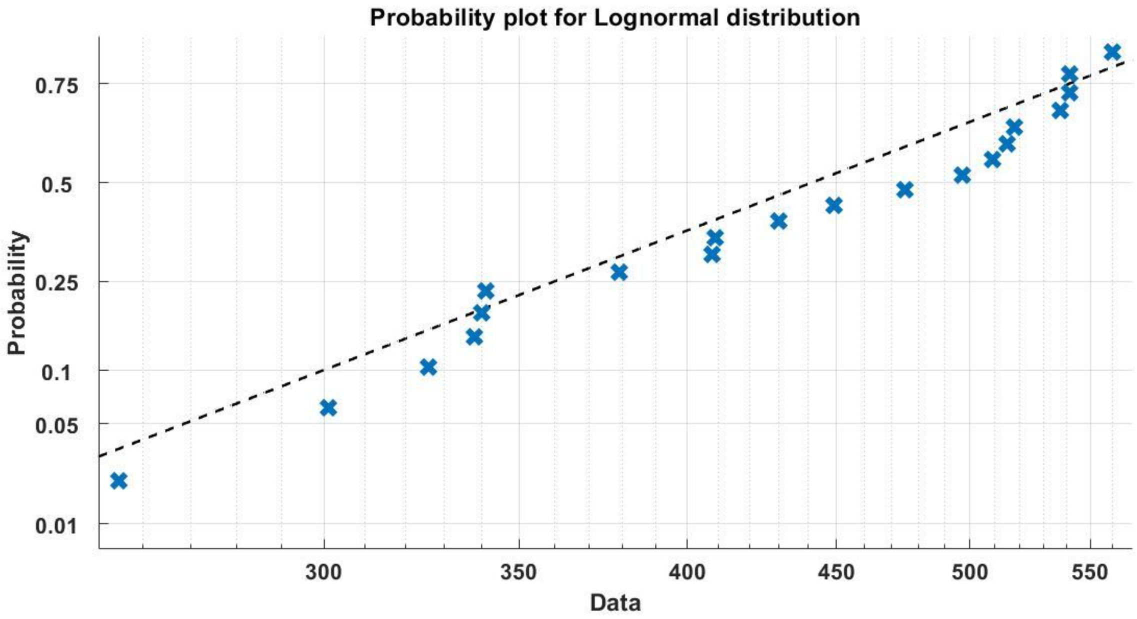

Fig. 4. The lognormal probability plot of the data in Table 4.

It is clear from the two plots that the Weibull and lognormal properly fit the observed data. Let us now run the two codes and compare the graphical observation with the analytical results, where the analytical approaches depend on the p-value of each distribution. As in the first example, the user must insert the following inputs to run the codes

```
data={255, 301, 326, 338, 340, 341, 379, 408, 409, 430, 449, 475, 497, 509,
515, 518, 537, 541, 541, 560};
t=593;
n=24;
M=10000;
sl=0.05;
```

and the outputs for the two distributions will be as follows:

Table 5: The CVM test results for Weibull and lognormal distributions of the Lithium-Ion Batteries life data provided in Table 4

Code I	**Code II**

The Cramer-von Mises Test Results for Weibull Distributio n The MLE of α= 5.21824 The MLE of β= 516.879 The test statistics of the observed data= 0.0648943 The simulated critical value of the test= 0. 155219 The simulated p-value= 0. 4646 Decision: 'Accept'	The Cramer-von Mises Test Results for Lognormal Distributio n The MLE of θ= 6.12912 The MLE of τ= 0.279582 The test statistics of the observed data= 0.0439397 The simulated critical value of the test= 0. 10126 The simulated p-value= 0. 4263 Decision: 'Accept'

Table 6: The AD test results for Weibull and lognormal distributions of the Lithium-Ion Batteries life data provided in Table 4

Code III	**Code IV**
The Anderson-Darling Test Results for Weibull Distributio n The MLE of α= 5.21824 The MLE of β= 516.879 The test statistics of the observed data= 0.418179	The Anderson-Darling Test Results for Lognormal Distributio n The MLE of θ= 6.12912 The MLE of τ= 0.279582 The test statistics of the observed data= 0.233322

`The simulated critical value of the test=`
`0. 810849`
 	
`The simulated p-value=`
`0. 4666`
 	
`Decision: 'Accept'`
	`The simulated critical value of the test=`
	`0. 546034`
	
	`The simulated p-value=`
	`0. 4852`
	
	`Decision: 'Accept'`

By looking at the p-values of both tests for the two distributions, we can see that CVM prefers the Weibull, whereas AD prefers the lognormal. This exactly agrees with the probability plots of the two distributions in Figs 3 and 4. In fact, as mentioned in the introduction, AD test gives more weight to the tails, whereas CVM focuses more on the center. We can notice from the two figures that the lognormal distribution catches more points on its left tail, whereas the Weibull better controls the center of the data. In this case more goodness-of-fit tests are needed to give the final decision, perhaps the Chi-square [6] or Lilliefors [23] goodness-of-fit tests as discussed in [17], which show that lognormal is a better model to the experimental data. In [17], the authors use simulation to generate the critical and p values of the CVM. Their results are different from what is presented in Table 5 because they consider the data as a complete sample although they deal with the data as censored when estimating the two distributions' parameters.

Example 4.3: (Solid-state Lighting) Solid-state lighting is a promising technology to save energy, reduce cost, transmit data, connect lighting with other building systems and essentially revolutionize our entire lighting infrastructure. The department of energy of the U.S estimates that solid-state lighting luminaire is expected to save energy cost by $ 250 billion over next 20 years and avoid 1,800 million metric tons of CO2 [24]. Reliability remains one of the challenges, hindering further proliferation of this technology, and there is a crucial need for lighting industry and research centers to understand the durability and failure models of the solid-state lighting luminaires, and to develop precise probability distributions of the failures. In this example, we will apply the two codes in testing the compliance of the censored solid-state lighting (SSL) luminaire experimental data of the hammer test of [24]. The hammer test described in [24], is an accelerated life test applied to 17 different commercial luminaires with test duration of 1470 hours. 12 out of 17 luminaires failed before that time as five unites still operating and their failure times are reported in Table 7. In [10] the authors give a brief but convenient description of the hammer test and conduct a comparison between the censored samples of Weibull and lognormal distributions based on the efficiency measure [22]. However, none of these works conducted analytical or even graphical goodness-of-fit tests to choose the most preferable models among the two proposed.

Table 7: The failure times, in hours, of the 12 luminaires described in Example 4.3

293	293	336	456	547
586	754	800	888	926
969	1176			

Just like Examples 4.1 and 4.2, the following are the inputs and the outputs of the two tests.

Inputs:

```
data={293,293,336,456,547,586,754,800,888,926,969,1176};

t=1470;

n=17;

M=10000;

sl=0.05;
```

Outputs:

Table 8: The CVM test results for Weibull and lognormal distributions of the Solid State life data provided in Table 7

Code I	**Code II**
The Cramer-von Mises Test Results for Weibull Distributio n	The Cramer-von Mises Test Results for Lognormal Distributio n

The MLE of α= 1.68709 The MLE of β= 1198.55 The test statistics of the observed data= 0.0418625 The simulated critical value of the test= 0. 0987321 The simulated p-value= 0. 4274 Decision: 'Accept'	The MLE of θ= 6.79907 The MLE of τ= 0.753565 The test statistics of the observed data= 0.0196125 The simulated critical value of the test= 0. 089893 The simulated p-value= 0. 8195 Decision: 'Accept'

Table 9: The AD test results for Weibull and lognormal distributions of the Solid State life data provided in Table 7

Code III	**Code IV**
The Anderson-Darling Test Results for Weibull Distributio n The MLE of α= 1.68709 The MLE of β= 1198.55 The test statistics of the observed data= 0.243018 The simulated critical value of the test= 0. 555128	The Anderson-Darling Test Results for Lognormal Distributio n The MLE of θ= 6.79907 The MLE of τ= 0.753565 The test statistics of the observed data= 0.141242 The simulated critical value of the test= 0. 494018

The simulated p-value= 0.4675 Decision: 'Accept'	The simulated p-value= 0.7479 Decision: 'Accept'

The two codes suggest that the best model is lognormal as it achieves the highest p-values for the two tests with significant difference. These results agree with the probability plots of the two distributions as illustrated in Figs 5 and 6, and with the conclusion of [10].

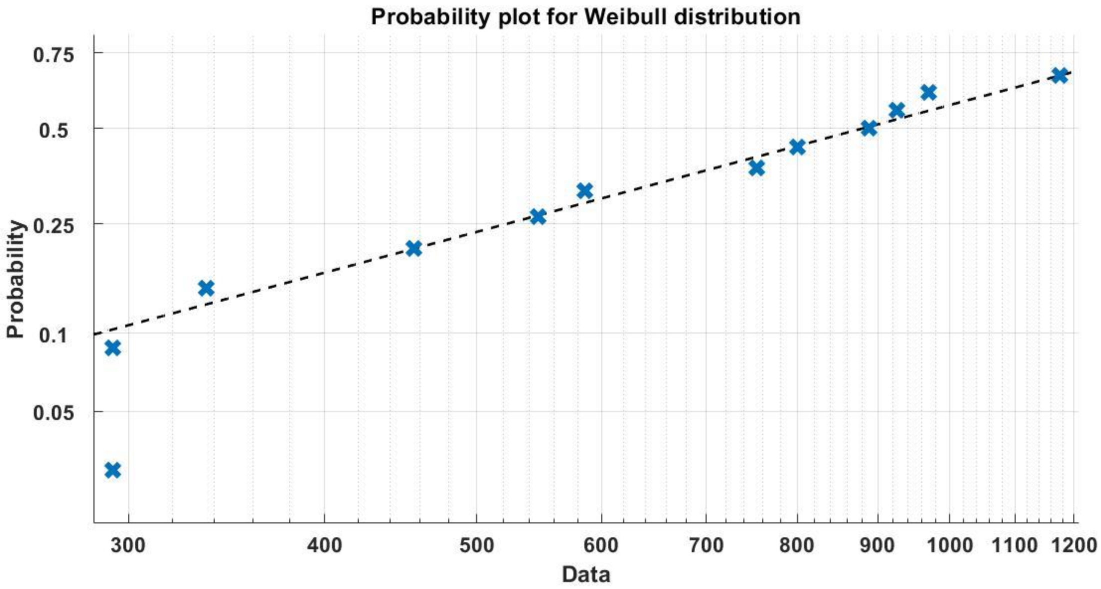

Fig. 5. The Weibull probability plot of the data in Table 7.

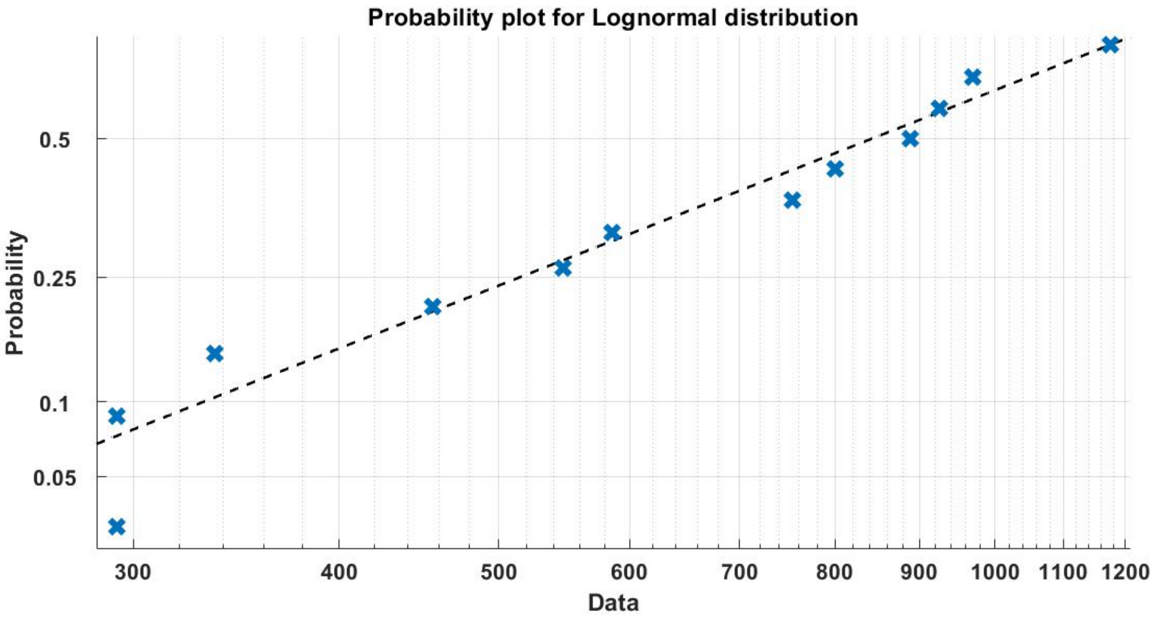

Fig. 6. The lognormal probability plot of the data in Table 7.

5. Conclusions and suggestions

This chapter has introduced Mathematica codes of two of the most powerful goodness-of-fit tests for censored data, the Cramer-von Mises (CVM) and Anderson-Darling (AD), for two of the most fundamental probability distributions in reliability studies, Weibull and lognormal. Such codes are crucial for practitioners and researchers in the field of reliability or related fields. The reason is that most of the experimental data are incomplete, and unfortunately, the available tests with built-in functions on most of the ready programs are only valid for complete data like MATLAB and Mathematica. The codes also provide the maximum likelihood estimates from censored data, which are not available on both packages. The type of censoring considered in this chapter is type I right censoring, the most commonly used type in applications. The codes used Monte Carlo simulations and designed on the basis of solid mathematical formulas published in reliable venues. The codes automatically provide the test statistics, critical value, and p-value of the test. The codes have been applied to test real data sets of lifetimes of energy and engineering items that were published before. The main disadvantage of the codes is that they are time consuming. When using a computer with normal speed, for Weibull distribution case, both CVM and AD codes need around 0.5 minute, when the number of simulations is 5000 for samples of size 10, to 5 minutes, for 20000 simulations with sample size 30, whereas for the lognormal distribution, the codes take less than these times.

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