Thermodynamics of the Bardeen Regular Black Hole *

M. Akbar**, Nema Salem, S. A. Hussein

Effat University, Jeddah 21478, Kingdom of Saudi Arabia

(Received 15 March 2012)

We deal with the thermodynamic properties of the Bardeen regular black hole with reference to their respective horizons. It is argued here that the expression of the heat capacity at horizons is positive in one parameter region and negative in the other, and between them the heat capacity diverges where the black hole undergoes the second-order phase transition.

PACS: 04.70.Dy, 97.60.Lf DOI: 10.1088/0256-307X/29/7/070401

The deep connection between gravity and thermodynamics has been studied extensively in literature[1] as a consequence to the finding that the four laws of the black hole thermodynamics could be derived from the Einstein field equations. It has been substantiated further with the Hawking discovery that the black hole can radiate thermal radiation[2] with a temperature proportional to the surface gravity and entropy proportional to the surface area of the black horizon. By resorting to Hawking's semi-classical treatment of the subject, it has been demonstrated that the temperature of the black hole would increase with the decrease of its mass.[3] Therefore, Bakenstein speculated that the entropy of the black hole decreases in time with the emission of Hawking radiation and hence it violates[4] the second law of thermodynamics. He further argues that the sum of the rate of change of entropies of black hole horizons and the entropy of the background universe has to be positive and is termed as the generalized second law (GSL) of thermodynamics.

It has been advocated further that the notions of temperature and entropy could not be limited only to the black hole horizons but they could also be extended[5] to the horizons of any spacetime geometry. This inspires that the profound relation between the laws of thermodynamics and gravity can be extended to any spacetime geometries with horizons. The first and the GSL[6−8] of the black hole has been studied widely in different spacetime geometries and also extended to the various gravity theories. In spite of the above thermodynamic analysis of the black hole, the statistical nature of the black hole thermodynamics remains obscure. However, the thermal stability of a black hole is determined by the sign of its heat capacity. For example, the heat capacity of the Schwarzschild black hole is always negative and is therefore thermodynamically unstable.[3] As for the Reissner–Nordstrom black hole, Kerr black hole, and Kerr–Newman black hole, their heat capacity is positive and negative depending on their parameter regions. It is, therefore, stated that a black hole is supposed to be thermodynamically unstable if its heat capacity is negative. However, every time the heat capacity changes sign in the parameter space, its heat capacity diverges during this process of change and subsequently undergoes a second order phase transition.[9,10] It is, nevertheless, possible to have a thermodynamically stable black hole that contains positive specific heat through a phase transition from thermal radiations.[11] In this context, a method of phase transition is established by Hawking and Page[12] through the demonstration of the transition between thermal AdS space and the Schwarzschild-AdS (SAdS) black hole.[13] In this Letter, we analyze the Bardeen black hole by discussing its thermodynamic aspects with a special reference to its phase transition caused by its heat capacity.

To begin with, we present the Bardeen regular black hole metric

$$ds^2 = -Fdt^2 + F^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \quad (1)$$

the mass function $M(r)$ in $F(r) = 1 - \frac{2M(r)}{r}$ represents arbitrary continuous function of variable r. With the following particular value of this mass function,

$$M(r) = \frac{mr^3}{(r^2+e^2)^{3/2}}, \quad (2)$$

the metric (1) reduces to the Bardeen regular black hole[14] metric. The m and e in the above equation represent the mass and the monopole charge of a self-gravitating magnetic field of a non-linear electrodynamics source, respectively. If one considers $e = 0$, the Bardeen black hole reduces to the Schwarzschild metric. The horizons of the Bardeen black hole are the roots of the function $F(r)$. These roots are obtained by setting

$$F(r) = 1 - \frac{2mr^2}{(r^2+e^2)^{3/2}} = 0. \quad (3)$$

*Supported by the Research Consultancy Institute (RCI), Effat University, Jeddah, Kingdom of Saudi Arabia.

**Corresponding author. Email: makbar@effatuniversity.edu.sa

Instead of finding the exact expressions of the roots of the above equation, $F(r) = 0$, the function $F(r)$ is explored here qualitatively in order to find the conditions on mass m and charge e at the horizons of the black hole. In order to find different conditions on mass m and charge e of the black hole, the roots of $F(r)$ can be categorized into three cases: (i) two distinct roots, (ii) repeated roots (the extreme case), and (iii) no real roots (naked singularity).

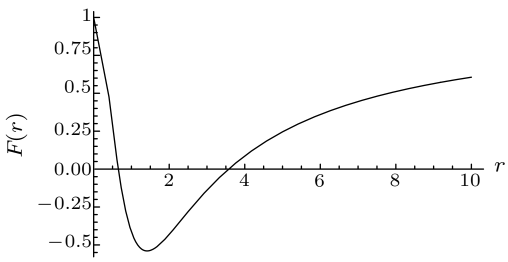

Fig. 1. For $F(r_{\min}) < 0$, the graph of $F(r)$ for $m = 2$ and $e = 1$ is shown, which cuts the r-axis at two distinct points $r = r_{\pm}$.

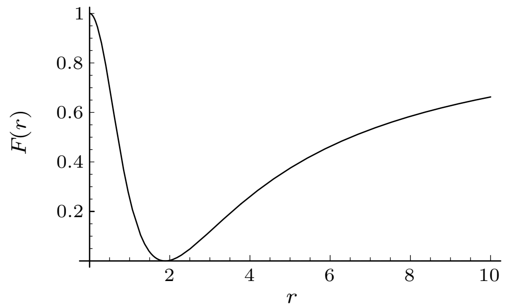

Fig. 2. For $F(r_{\min}) = 0$, the graph of F(r) for $m = \sqrt{3}$ and $e = \frac{4}{3}$ is shown which cuts the r-axis at one point for which $r = r_+ = r_-$.

To carry out our analysis, let us first calculate the extremal points of the function $F(r)$ by setting $dF/dr = 0$, which corresponds to $r_{\min} = \sqrt{2}e$, excluding the negative root as it gives $r < 0$. This extremal point is a minima because of $d^2F/dr^2 > 0$ at $r = r_{\min}$. Hence, the minimum value of the function at $r = r_{\min}$ is

$$F(r_{\min}) = 1 - \frac{4m}{3\sqrt{3}e}. \tag{4}$$

Moreover, $F(r) \to 1$ for both $r \to 0$ and $r \to \infty$ as shown in Fig. 1.

It is evident from Fig. 1 that the Bardeen black has two horizons, inner for $r = r_-$ and outer for $r = r_+$, implying $F(r_{\min}) < 0$, which admits a condition $3\sqrt{3}e < 4m$. This is, nonetheless, an already established condition for the Bardeen black hole. Hence, the graph of the function $F(r)$ intersects the r-axis at two points, $r = r_{\pm}$ leaving with one minimum point at $r_{\min} = \sqrt{2}e$ below the r-axis. For the extreme Bardeen black hole we resort to $F(r_{\min}) = 0$, which exhibits the condition, $3\sqrt{3}e = 4m$ as shown in Fig. 2.

In this case we have two real repeated roots, for which $r = r_+ = r_-$. The third case directs the naked Bardeen black hole, in which $F(r_{\min}) > 0$ and admits $3\sqrt{3}e > 4m$. In this case, no real roots exist and the graph of $F(r)$ does not intersect the r-axis, as shown in Fig. 3.

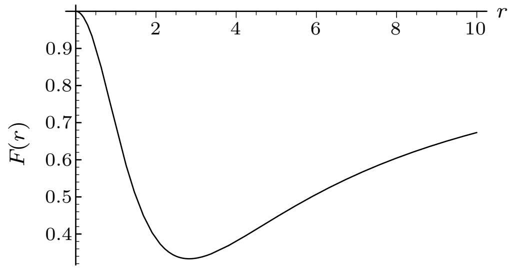

Fig. 3. For $F(r_{\min}) > 0$, the graph of $F(r)$ for $m = \sqrt{3}$ and $e = 2$ is shown, which does not cut the r-axis.

Ricci scalar R of the bardeen black is given by

$$R = \frac{6me^2(-r^2 + 4e^2)}{(r^2 + e^2)^{\frac{7}{2}}}. \tag{5}$$

It is important to note that (i) Ricci scalar vanishes identically at $r = 2e = \sqrt{2}\, r_{\min}$, admitting Ricci flat geometry of the Bardeen black hole, (ii) it is regular at $r = r_{\pm}$, and (iii) Ricci scalar also vanishes when $r \to \infty$.

The heat capacity is an important measure to study the thermodynamic properties of a black hole. Before we discuss the heat capacity of the Bardeen black hole, we must evaluate thermal quantities associated with the horizon at $r = r_+$. The temperature T of the Bardeen black hole can be determined by using the relation $T = \frac{1}{4\pi}\frac{dF(r)}{dr}|_{r=r_+}$, which admits

$$T = \frac{mr_+(r_+^2 - 2e^2)}{2\pi(r_+^2 + e^2)^{5/2}}, \tag{6}$$

where the mass m of the Bardeen black hole can be defined in terms of outer horizon as

$$m = \frac{(r_+^2 + e^2)^{3/2}}{2r_+^2}. \tag{7}$$

The temperature T of the Bardeen black is positive, zero and negative, accordingly $r_+ > \sqrt{2}e$, $r_+ = \sqrt{2}e$, and $r_+ < \sqrt{2}e$, respectively. The temperature T at the horizon can be rewritten in terms of r_+ as

$$T = \frac{r_+^2 - 2e^2}{4\pi r_+(r_+^2 + e^2)}. \tag{8}$$

The heat capacity of the Bardeen black hole can be evaluated at the horizon $r = r_+$ by using the relation $C_e = (\frac{\partial m}{\partial T})_e$ at the constant charge e. The partial derivative of mass m and temperature T with respect to r_+ is given, respectively, as

$$\frac{\partial m}{\partial r_+} = \frac{(r_+^2 + e^2)^{1/2}(r_+^2 - 2e^2)}{2r_+^3}, \tag{9}$$

$$\frac{\partial T}{\partial r_+} = \frac{2e^4 + 7r_+^2e^2 - r_+^4}{4\pi r_+^2(r_+^2 + e^2)^2}. \tag{10}$$

By making use of the above equations, the heat capacity $C_e = \frac{\partial M}{\partial r_+}\frac{\partial r_+}{\partial T}$, when evaluated at the horizon, admits

$$C_e = \frac{2\pi(r_+^2 + e^2)^{5/2}(r_+^2 - 2e^2)}{r_+(2e^4 + 7e^2r_+^2 - r_+^4)}. \tag{11}$$

It follows from the above expression (11) that the sign of the heat capacity of the Bardeen black hole mainly depends on two terms $(r_+^2 - 2e^2)$ and $(2e^4 + 7e^2r_+^2 - r_+^4)$. Based on the sign of the above terms, one can determine whether the heat capacity is negative, zero, or positive. It is important to first analyze the term, $(2e^4 + 7e^2r_+^2 - r_+^4)$ by defining a new variable $R = r/e$ to evaluate its nature. Substituting $R = r/e$ in this term, one can further analyze it by considering the function

$$f(R) = 2 + 7R^2 - R^4. \tag{12}$$

It becomes necessary here to first workout the extremal points of the above function (12) by making use of $df/dR = 0$, which admits two points $R = 0$ and $R = \sqrt{7/2}$. The first point, $R = 0$, admits minima, for which $f(R_{\min} = 0) = 2$, while the second point, $R = \sqrt{7/2}$, corresponds to maxima admitting the maximum value, $f(R_{max}) = 57/4$. On the other hand, $f(R)$ is zero at $R = \sqrt{7/2 + \sqrt{57/4}}$. Hence, from the above analysis, $f(R)$ is positive in the interval, $0 \le R < \sqrt{7/2 + \sqrt{57/4}}$ or alternatively $0 \le r_+ < (\sqrt{7/2 + \sqrt{57/4}})e$ and is negative for $R > \sqrt{7/2 + \sqrt{57/4}}$ or alternatively $r_+ > (\sqrt{7/2 + \sqrt{57/4}})e$, as shown in Fig. 4.

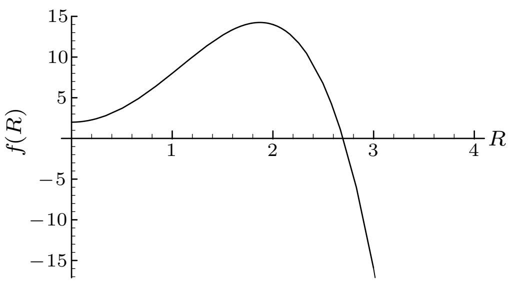

Fig. 4. Graph of $f(R)$ is shown, which has the maximum value at $R = \sqrt{7/2}$ and cuts the R-axis.

On the other hand, the second term $(r_+^2 - 2e^2)$ is negative, zero or positive, accordingly $0 < r_+ < \sqrt{2}e$, $r_+ = \sqrt{2}e$ or $r_+ > \sqrt{2}e$. By combining the analysis of these two terms altogether, one can find out the intervals, $0 < r_+ < \sqrt{2}e$, $r_+ = \sqrt{2}e$, or $\sqrt{2}e < r_+ < (\sqrt{7/2 + \sqrt{57/4}})e$, for which the heat capacity is negative, zero, or positive respectively, while it diverges at $r_+ = (\sqrt{7/2 + \sqrt{57/4}})e$. In fact, the change of sign in the heat capacity reflects the fundamental change in the stability properties of the thermal system. A region, in which the heat capacity is negative, corresponds to a region of instability, whereas a region, in which the heat capacity is positive, reflects a stability region. It is obvious from the above analysis that the interval, $0 < r_+ < \sqrt{2}e$ is the instability region in which the heat capacity is negative, whereas the interval $\sqrt{2}e < r_+ < (\sqrt{7/2 + \sqrt{57/4}})e$ is the stability region with the positive heat capacity. On the other hand, the heat capacity of the Bardeen black hole diverges at point $r_+ = (\sqrt{7/2 + \sqrt{57/4}})e$, which corresponds to the second-order phase transition. At this stage, one can rewrite the expression of the heat capacity in terms of $\frac{r_+}{e} = R$ to visualize its graphical behavior as

$$C(R) = \frac{2\pi e^2(R^2 + 1)^{5/2}(R^2 - 2)}{R(2 + 7R^2 - R^4)}. \tag{13}$$

The graphical picture of the heat capacity (13) is shown in Fig. 5.

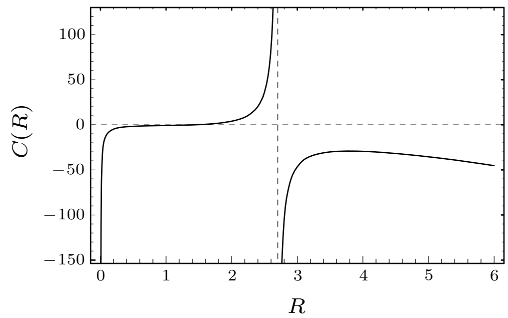

Fig. 5. Graph of heat capacity shows that it contains instability, stability regions and undergoes phase transition.

In summary, we have studied the thermodynamic properties of the regular Bardeen Black hole with reference to their respective horizons. According to the nature of the roots, we characterize three cases for the Bardeen regular black hole. We derive the expressions for temperature, electric potential and heat capacity. It is found that the temperature of the Bardeen black hole is negative, zero and positive accordingly $0 < r_+ < \sqrt{2}e$, $r_+ = \sqrt{2}e$ and $r_+ > \sqrt{2}e$. According to the behavior of heat capacity, we study the regions of stabilities of the black hole. It is found that the interval $0 < r_+ < \sqrt{2}e$ is the instability region for which the heat capacity is negative, while the interval $\sqrt{2}e < r_+ < (\sqrt{7/2 + \sqrt{57/4}})e$ is the stability region bearing positive heat capacity. The heat capacity is zero at $r_+ = \sqrt{2}e$ which corresponds to the extreme Bardeen black hole. On the other hand, the heat capacity of the Bardeen black hole diverges at a point $r_+ = (\sqrt{7/2 + \sqrt{57/4}})e$, which indicates that the Bardeen regular black hole undergoes second-order phase transition.

The authors thank Haifa Jamal Al-Lail, Houria Oudghiri, Azza Abou Zeid and M. Khalid for their continuous support for research activities at Effat University. We would like to thank referees for their useful comments.

References

[1] Bardeen J M et al 1973 *Commun. Math. Phys.* **31** 161
Akbar M 2007 *Chin. Phys. Lett.* **24** 1158
Akbar M andCai R G 2007 *Phys. Lett.* B **648** 243
Kolekar S, Kothawala D and Padmanabhan T 2011 arXiv:1111.0973 [gr-qc]
Miao R X, Miao Liand Miao Y G 2011 *J. Cosmol. Astropart. Phys.* **1111** 033
Akbar M and Cai R G 2007 *Phys. Rev.* D **75** 084003
Akbar M and Cai R G 2006 *Phys. Lett.* B **635** 7
Cai R G and Ohta N 2010 *Phys. Rev.* D **81** 084061
Cai R G, Li-Ming Cao and Ohta N 2009 *Phys. Lett.* B **679** 504
Padmanabhan T 2010 *Rept. Prog. Phys.* **73** 046901
Kothawala T and Padmanabhan T 2009 *Phys. Rev.* D **79** 104020 and references therein

[2] Hawking S W 1975 *Commun. Math. Phys.* **43** 199

[3] Shen J, Cai R G, Wang B and Su R K 2007 *Int. J. Mod. Phys.* A **22** 11
Ashtekar A, Baez J, Corichi A and Krasnov K 1998 *Phys. Rev. Lett.* **80** 904
Jamil M and Qadir A 2011 *Gen. Rel. Grav.* **43** 1069

[4] Bekenstein J D 1973 *Phys. Rev.* D **7** 2333
Bekenstein J D 1974 *Phys. Rev.* D **9** 3292

[5] Frolov A V and Kofman L 2003 *J. Cosmol. Astropart. Phys.* **0305** 009
Danielsson U K 2005 *Phys. Rev.* D **71** 023516
Bousso R 2005 *Phys. Rev.* D **71** 064024

[6] Bokhari A H and Akbar M 2010 *Int. J. Mod. Phys.* **142** 1656
Jamil M and Akbar M 2010 arXiv:1005.3444 [gr-qc]
Akbar M 2009 *Int. J. Theor. Phys.* **48** 2665
Akbar M 2008 *Chin. Phys. Lett.* **25** 4199
Mohseni Sadjadi H and Jamil M 2010 arXiv:1002.3588 [gr-qc]
Radicella N and Pavon D 2010 *Phys. Lett.* B **691** 121
Karami K and Ghaffari S 2010 *Phys. Lett.* B **688** 125
Barrow J D 1998 *Nucl. Phys.* B **310** 743
Giovannini M 1999 *Phys. Rev.* D **59** 121301
Zimdahl Wand and Pavon D 2000 *Phys. Rev.* D **61** 108301
Setare M R2006 *Phys. Lett.* B **641** 130
Wu S F, Wang B, Yang G H and Zhang P M 2008 arXiv:0801.2688 [hep-th]
Bekenstein J D and Nuovo 1972 *Cim. Lett.* **4** 737
Bekenstein J D and Nuovo 1974 *Phys. Rev.* D **9** 3292

[7] Rehman F, Salahuddin and Akbar M 2011 *Chin. Phys. Lett.* **28** 070403
Hod S 2011 *Phys. Lett.* B **700** 75
Bandyopadhyay T and Debnath U 2011 *Phys. Lett.* B **704** 95
Setare M R and Sheykhi A 2010 *Int. J. Mod. Phys.* D **19** 1205

[8] Karami K, Chaffari S and Soltanzadeh M M 2010 *Class. Quant. Grav.* **27** 205021
Mohseni Sadjadi H 2010 *Europhys. Lett.* **92** 50014
Nouicer K 2011 *Int. J. Mod. Phys.* D **20** 233
He S and Zhang H 2007 *J. High Energy Phys.* **0712** 052

[9] Hut P 1977 *Mon. Not. R. Astron Soc.* **180** 379
Davies P C W 1977 *Proc. Roy. Soc. Lond.* A **353** 499
Davies P C W 1977 *Rep. Prog. Phys.* **41** 1313
Davies P C W 1989 *Class. Quant. Grav.* **6** 1909

[10] Sokolowski L M and Mazur P 1980 *J. Phys.* A **13** 1113
Lau Y K 1994 *Phys. Lett.* A **186** 41
Gros D J, Perry M J andYaffe L G 1982 *Phys. Rev.* D **25** 330
Lousto C O 1995 *Phys. Rev.* D **51** 1733

[11] Myung Y S 2008 *Phys. Rev.* D **77** 104007

[12] Hawking S W and Page D N 1983 *Commun. Math. Phys.* **87** 577

[13] Brown J D, Creighton J and Mann R B 1994 *Phys. Rev.* D **50** 6394

[14] Sharif M and Javed W 2010 *J. Korean Phys. Soc.* **57** 217
Sharif M and Javed W 2011 *Can. J. Phys.* **89** 1027